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THE EFFECT OF A LOCAL INCREASE IN IMPURITY CONCENTRATION IN ONE-DIMENSIONAL HYDRODYNAMICS†

V. A. GORODTSOV

Moscow

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The system of non-linear equations of one-dimensional convective diffusion of a passive impurity in the case of a flow of fluid described by the model Burgers equation is solved using Cole–Hopf and Darboux transformations. For equal kinetic coefficients of the fluid its solution is reduced to solving linear heat-conduction equations. For integer Prandtl numbers, differing from unity, this reduction is successful for flows of uniformly moving shock waves, smoothed by the viscosity effect. Its possibility is closely related to the factorizability of second-order differential operators in this case, their decomposability into the product of first-order operators, and the additional internal symmetry (supersymmetry) of the problem. The interaction of shock waves and the impurity solitons they transfer has an absolutely inelastic character. A local increase in the impurity concentration is a result of the merging of impurity solitons. It is pointed out that, for certain equations of state, the reduction of the solutions of the non-linear convective diffusion equations to the solution of linear heat-conduction equations is also possible for an active impurity (when the kinetic coefficients of the fluid are equal). © 2000 Elsevier Science Ltd. All rights reserved.

1. THE CONVECTION OF A PASSIVE IMPURITY FOR UNIT PRANDTL NUMBER

In Kovalevskaya's papers, devoted to solving the problem of the motion of a solid in a gravity force field, fixed at a single point [1, 2], a relation between the integrability of non-linear ordinary differential equations and the analytical properties of their solutions was used for the first time. This relation was then employed by Painlevé and co-workers to classify and exhaustively analyse the solutions of second-order non-linear ordinary differential equations [3, 4]. In the Kovalevskaya–Painlevé method the integrability of ordinary differential equations is linked to the possibility of representing the solution in the form of Laurent series with a finite principal polar part

$$(t-t_0)^{-n}\sum_{m=0}^{\infty}a_m(t-t_0)^m$$

with a sufficient number of arbitrary parameters of the expansion (the latter is determined by the order of the equation).

The Kovalevskaya–Painlevé method was extended directly to analyse non-linear partial differential equations in [5]. To do this one must replace the difference $t - t_0$ in the case, for example, of two independent variables, by a function $\varepsilon(x, t)$ and use a Laurent-type expansion about the singular manifold $\varepsilon(x, t) = 0, \partial_t\varepsilon \neq 0, \partial_x\varepsilon \neq 0$, taking into account the coefficients of the expansion of the functions x and t also

$$u(x,t)=\sum_n u_n\varepsilon^{n-\alpha},\quad u_n=u_n(x,t),\quad \varepsilon=\varepsilon(x,t),\quad u_n\big|_{n<0}=0$$

If after substituting this expansion of the solution into the equation, the exponent α turns out to be an integer (a pole singularity) and the recurrence relation for the coefficients contains a sufficient number of arbitrary functions $u_n(x, t)$ and $\varepsilon(x, t)$, corresponding to the order of the equation, one can speak of the Kovalevskaya–Painlevé test in the Weiss–Tabor–Carnevale form for the equation in question as being satisfied. This turns out to be a sufficient indicator of its integrability [5].

In the case of the model equation of one-dimensional hydrodynamics, not containing the pressure, which is usually called the Burgers equation [6, 7]

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = 0,\quad u = u(x,t) \tag{1.1}$$

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the Kovalevskaya–Painlevé test is satisfied [5] and the solution of the equation can be represented by the following Laurent series

$$u = -2\nu \frac{\varepsilon_x}{\varepsilon} + \sum_{n=0}^{\infty} u_{n+1}\varepsilon^n, \quad u_n = u_n(x,t), \quad \varepsilon = \varepsilon(x,t) \tag{1.2}$$

In this case two functions can be chosen arbitrarily from those indicated $\varepsilon(x, t)$ and $u_2(x, t)$ (this arbitrariness corresponds to the arbitrariness of the general solution of a second-order equation), while the coefficient function $u_1(x, t)$ satisfies the initial Burgers equation. If the regular part of the expansion is neglected, in the final analysis a form of the well-known Cole–Hopf replacement [8, 9] is established, by means of which the solution of a non-linear equation can be reduced to the solution of the linear heat-conduction equation.

The description of non-dimensional convection of a passive impurity requires the simultaneous analysis of the non-linear equation of motion (1.1) with equations of the form

$$\partial_t c + \partial_x(uc) - \chi\partial_x^2 c = 0 \tag{1.3}$$

$$\partial_t k + u\partial_x k - \chi\partial_x^2 k = 0 \tag{1.4}$$

where Eq. (1.3) is obtained from the last equation by simple differentiation, $c = \partial_x k$. The system of equations (1.1), (1.4) also permits of the Kovalevskaya–Painlevé test with Laurent expansion (1.2) for the velocity and a similar expansion for the concentration only in the special case of unit Prandtl number $\sigma \equiv \nu/\chi = 1$ [10]. In this case a form of the generalized Cole–Hopf transformation is established, namely

$$u = -2\nu\partial_x \ln\varepsilon, \quad k = \varphi/\varepsilon$$

by means of which a system of linked non-linear equations in the case of different kinetic coefficients can be reduced to separate linear heat-conduction equations

$$\partial_t\varepsilon = \nu\partial_x^2\varepsilon, \quad \partial_t\varphi = \nu\partial_x^2\varphi \tag{1.5}$$

Their simplest solutions correspond to the physically interesting solutions of the initial non-linear equations. For example, the exponential in the sum with the unit solution for ε

$$\varepsilon = 1 + \exp(-\theta), \quad \theta \equiv \frac{u_0}{\nu}(x - x_0 - u_0 t) \tag{1.6}$$

corresponds to a single uniformly moving shock wave in the form of a sudden drop in the velocity with a smooth transition region of width $\sim\nu/u_0$

$$u = \frac{2u_0}{1+\exp\theta} = u_0\left(1 - \operatorname{th}\frac{\theta}{2}\right) \tag{1.7}$$

The interaction between several shock waves is described by the sum of the exponential functions for ε [11, 12]

$$\varepsilon = 1 + \sum_i \exp[-(\theta_i - \theta_{0i})], \quad \theta_i \equiv \frac{u_i}{2\nu}\left(x - \frac{u_i}{2}t\right)$$

In particular, the similar sum with two exponential functions ($u_2 > u_1 > 0$)

$$\varepsilon = 1 + \exp(-\theta_1) + \exp(-\theta_2 + \theta_0), \quad u = \frac{u_1\exp(-\theta_1) + u_2\exp(-\theta_2 + \theta_0)}{1 + \exp(-\theta_1) + \exp(-\theta_2 + \theta_0)}$$

describes the merging of a pair of shock waves, of which the one possessing the greater velocity $(u_2 + u_1)/2$ overtakes the other, having a velocity $u_1/2$, at the instant of time $2x_0/(u_2 - u_1)$, and again the wave formed as a result of the interaction moves further with a velocity $u_2/2$.

Convection of the initially localized passive impurity in the field of the uniformly moving shock wave, described by Eq. (1.3), is non-stationary, but the impurity distribution asymptotically acquires a simplified form which propagates steadily together with the shock wave of the soliton-like distribution [10]

$$c(x,t) \sim \operatorname{sech}^2 \frac{u_0}{2\nu}(x - u_0 t)$$

These distributions correspond completely in form with the solitons of the Korteweg–de Vries equation. However, unlike these well-known solitons, the "passive impurity solitons" discussed interact with one another absolutely inelastically in accordance with the inelastic character of the interaction of the shock waves transporting them. Finally, concentration of the impurity into a single overall soliton occurs.

2. ANALYSIS OF THE ALGEBRAIC STRUCTURE AND THE SOLUTIONS OF THE PASSIVE IMPURITY CONVECTION EQUATIONS

In the case of arbitrary Prandtl numbers the system of non-linear equations discussed can be reduced, using a similar generalized Cole–Hopf transformation of the form

$$u = -2\nu\frac{\partial_x \varepsilon}{\varepsilon}, \quad c = \frac{\psi}{\varepsilon^\sigma}, \quad k = \frac{\varphi}{\varepsilon^\sigma} \tag{2.1}$$

$$\psi = \hat{D}\varphi, \quad \hat{D}\ldots \equiv \left(\partial_x + \sigma\frac{u}{2\nu}\right)\ldots = \varepsilon^\sigma \partial_x \varepsilon^{-\sigma}\ldots \tag{2.2}$$

to three second-order linear differential equations

$$(\partial_t - \nu\partial_x^2)\varepsilon, \quad \hat{B}_+\psi = 0, \quad \hat{B}_-\varphi = 0 \tag{2.3}$$

$$\hat{B}_\pm \equiv \partial_t - \frac{\nu}{\sigma}\partial_x^2 + \frac{\sigma \pm 1}{2}u_x \tag{2.4}$$

However, the last two of these are linked to the first (the case $\sigma = 1$, when the equation for φ reduces to a separate diffusion equation is again anomalous here).

The Darboux operator appears in this generalized transformation. Its distinguishing feature is that it transforms the solution of a second-order linear differential equation into the solution of another equation of the same form (with a difference in the operators only in the coefficient functions). This role is clear from (2.2)–(2.4) and the directly verified commutation operator relation (the "interweaving relation")

$$\hat{B}_+\hat{D} = \hat{D}\hat{B}_- \tag{2.5}$$

Here it is more convenient to use dimensionless variables. In dimensionless variables

$$\eta = \frac{u_0}{2\nu}x, \quad \tau = \frac{u_0^2}{4\nu\sigma}t, \quad v = \frac{u}{u_0} \tag{2.6}$$

formed using the characteristic velocity u_0 and the viscosity coefficient, the system of the initial non-linear equations can be rewritten as follows:

$$\sigma^{-1}\partial_\tau \upsilon + 2\upsilon\partial_\eta \upsilon = \partial_\eta^2 \upsilon, \quad \partial_\tau c + 2\sigma\partial_\eta(\upsilon c) = \partial_\eta^2 c, \quad \partial_\tau k + 2\sigma\upsilon\partial_\eta k = \partial_\eta^2 k \tag{2.7}$$

where, in view of the linearity of the pair of equations for the passive impurity concentration its scale can be chosen arbitrarily. The generalized Cole–Hopf transformation (2.1), (2.2) and the transformed equations (2.3) and (2.4) in dimensionless form will appear as

$$\begin{gathered}
\upsilon = -\partial_\eta \ln\varepsilon, \quad c = \frac{\psi^{(\sigma)}}{\varepsilon^\sigma}, \quad k = \frac{\varphi^{(\sigma)}}{\varepsilon^\sigma}, \quad \psi^{(\sigma)} = \frac{u_0}{2\nu}\hat{D}_\sigma\varphi^{(\sigma)}, \quad \hat{D}_\sigma = \partial_\eta + \sigma\upsilon = \frac{2\nu}{u_0}\hat{D} \\
(\sigma^{-1}\partial_\tau - \partial_\eta^2)\varepsilon = 0, \quad \hat{L}_\sigma\psi^{(\sigma)} = 0, \quad \hat{L}_{\sigma-1}\varphi^{(\sigma)} = 0 \\
\hat{L}_\sigma \equiv \partial_\tau - \partial_\eta^2 + \sigma(\sigma+1)\upsilon_\eta = \frac{4\nu\sigma}{u_0^2}\hat{B}_+, \quad \hat{B}_- = \frac{u_0^2}{4\nu\sigma}\hat{L}_{\sigma-1}
\end{gathered} \tag{2.8}$$

Commutation relation (2.5) is rewritten in this case as the "ladder" relation

$$\hat{L}_\sigma \hat{D}_\sigma = \hat{D}_\sigma \hat{L}_{\sigma-1} \tag{2.9}$$

The dimensionless Darboux operator $\hat{D}_\sigma$, as can be seen from here, will be a ladder operator, commutation with which enables the value of the parameter σ to be changed by unity in the unique combination $\sigma(\sigma+1)$ which explicitly occurs in the operator $\hat{L}_\sigma$. If it is a "lowering operator", then in view of the formal symmetry property $\hat{L}_{-\sigma} = \hat{L}_{\sigma-1}$ (by virtue of thermodynamic limitations, negative Prandtl numbers are impossible), the Darboux operator $\hat{D}_\sigma$ will, by (2.9), be a "raising operator". However, the explicit combination $\sigma(\sigma+1)$ does not, in general, exhaust the dependence of the operator discussed on the Prandtl number. Generally speaking, the dimensionless velocity υ, which also occurs in it, implicitly depends on the Prandtl number, as is clear from the first equation of (2.7). In view of this, the chain of relations with different Prandtl numbers, following from (2.9), turns out to be open. Its closure occurs in the special case of a steady velocity field since, when $\partial_\tau \upsilon = 0$ the distribution of the dimensionless velocity ceases to depend on the Prandtl number (see (2.7)). Finally then, on the basis of (2.9), we obtain the following linked sequence of operator relations

$$\hat{L}_\sigma \hat{D}_\sigma \hat{D}_{\sigma-1} \ldots \hat{D}_{\sigma-n+1} = \hat{D}_\sigma \hat{D}_{\sigma-1} \ldots \hat{D}_{\sigma-n+1} \hat{L}_{\sigma-n} \tag{2.10}$$

and by repeated use of the lowering or raising operators we can achieve any integer change in the Prandtl number in the equations.

In particular, for integer Prandtl numbers the last sections in the reduction change will be operators with $\sigma = 1$ and $\sigma = 0$, so that the problem is reduced to a simple problem for the heat-conduction equation, and the following representation of the solution is possible

$$\hat{L}_\sigma \psi^{(\sigma)} = 0, \quad \psi^{(\sigma)} = \hat{D}_\sigma \ldots \hat{D}_1 \psi^{(0)}, \quad \hat{L}_0 \varphi^{(1)} = 0, \quad \hat{L}_0 = \partial_\tau - \partial_\eta^2, \quad \psi^{(0)} \equiv \frac{u_0}{2\nu} \varphi^{(1)} \tag{2.11}$$

The ladder form of the connection between the solutions with different Prandtl numbers obtained here turns out in fact to be a reflection of the fact that the Darboux operator when $\upsilon_\tau = 0$ is a root factorization operator of the steady part of the operator $\hat{L}_\sigma$ (reduction of second-order operators to the product of first-order operators), and commutes with the operator

$$\partial_\tau + \hat{D}_\sigma \hat{D}_\sigma^+ = \hat{L}_\sigma + \sigma^2 C, \quad \partial_\tau + \hat{D}_\sigma^+ \hat{D}_\sigma = \hat{L}_{\sigma-1} + \sigma^2 C; \quad \hat{D}_\sigma^+ = -\hat{D}_{-\sigma}$$
$$\hat{L}_\sigma \hat{D}_\sigma \varphi = (\partial_\tau + \hat{D}_\sigma \hat{D}_\sigma^+ - \sigma^2 C)\hat{D}_\sigma \varphi = \hat{D}_\sigma (\partial_\tau + \hat{D}_\sigma^+ \hat{D}_\sigma - \sigma^2 C)\varphi = \hat{D}_\sigma \hat{L}_{\sigma-1} \varphi$$

The superscript "plus" denotes a conjugate Darboux differential operator, and for transformations a corollary of the equation of motion $\upsilon^2 = \upsilon_n + C$ when $\upsilon_\tau = 0$ is used for transformations (for a solution of the form (2.13) the constant C becomes unity). Hence, factorization of the operators leads to property (2.9) [13, 14]. In turn, the possibility of factorization is connected with the additional internal symmetry of the problem [15,16].

On changing from a fixed to a uniformly moving system of coordinates, the non-linear hydrodynamic equations (1.1), (1.3) and (1.4) remain unchanged in accordance with the invariability of the impurity concentration fields and the change in the velocity field to an additive constant (Galilean invariance)

$$t' = t, \qquad x' = x - u_0 t \tag{2.12}$$
$$u'(x',t') = u(x,t) - u_0, \quad c'(x',t') = c(x,t), \quad k'(x',t') = k(x,t)$$

Because of this the solution in the form of a steady far-off bounded flow reduces to a uniformly moving shock wave. In a system of coordinates moving together with this wave, the velocity distribution in dimensionless variables (we omit the primes) has the form (compare with the form in dimensional variables in a laboratory system of coordinates (1.6), (1.7))

$$\upsilon = -\operatorname{th}\xi, \quad \xi = \eta - 2\sigma\tau = \frac{u_0}{2\nu}(x - u_0 t) \tag{2.13}$$

The new unknown quantities ε, ψ, φ, introduced by the generalized Cole–Hopf transformation, are not Galilean invariant. In accordance with the invariability of the concentration and the velocity field transformation (2.12) they vary as follows on changing to a moving system of coordinates

$$\begin{aligned} \varepsilon'(x',t') &= \varepsilon(x,t)\exp\left(\frac{u_0}{2\nu}x\right)f(t) \\ \psi'(x',t') &= \psi(x,t)\exp\left(\frac{u_0}{2\nu}\sigma x\right)f^{\sigma}(t) \\ \varphi'(x',t') &= \varphi(x,t)\exp\left(\frac{u_0}{2\nu}\sigma x\right)f^{\sigma}(t) \end{aligned} \tag{2.14}$$

The form of the diffusion equation for the function $\varepsilon(x, t)$ in (2.3) will not change when such a change of variables is made if we choose the arbitrary function of time $f(t)$ in the form

$$f(t) = C\exp\left(-\frac{u_0^2}{4\nu}t\right) \tag{2.15}$$

Here the arbitrariness of the constant C remains. In this case Eqs (2.3) for the two other functions $\psi(x, t)$, $\varphi(x, t)$ remain invariant automatically.

A reflection of these transformation properties of the auxiliary functions is, in particular, the fact that the following time-dependent function will correspond to the steady velocity distribution (2.13)

$$\varepsilon(\xi,\tau) = \operatorname{ch}\xi\exp(\sigma\tau)$$

Here, using the amplitude arbitrariness of (2.15) we make a simplifying choice of the arbitrary constant $C = 1/2$.

The linear equations considered and the connections between their solutions for integer Prandtl numbers can now be rewritten in a system of coordinates moving together with the shock wave as follows:

$$\hat{L}_\sigma\psi^{(\sigma)} = 0: \quad (\partial_\tau - \partial_\xi^2 - \sigma(\sigma+1)\operatorname{sech}^2\xi)\psi^{(\sigma)} = 0$$

$$\psi^{(\sigma)} = \hat{D}_\sigma\psi^{(\sigma-1)}: \quad \psi^{(\sigma)} = (\partial_\xi - \sigma\operatorname{th}\xi)\psi^{(\sigma-1)} = \operatorname{ch}^\sigma\xi\partial_\xi\frac{\psi^{(\sigma-1)}}{\operatorname{ch}^\sigma\xi}$$

$$\psi^{(\sigma)} = \operatorname{ch}^{\sigma+1}\xi\left(\frac{\partial}{\partial\operatorname{sh}\xi}\right)^\sigma\frac{\psi^{(0)}}{\operatorname{ch}\xi} = \operatorname{ch}^{\sigma+1}\xi\left(\frac{\partial}{\partial\operatorname{sh}\xi}\right)^{\sigma+1}\chi, \quad \psi^{(0)} = \hat{D}_0\chi = \partial_\xi\chi$$

If we confine ourselves to analysing the fundamental solution of the initial problem for the convective diffusion equation (1.3) in the velocity field of the shock wave, i.e. the initial problem with localized initial impurity distribution

$$c|_{t=0} = \delta(x - x_0) = \frac{u_0}{2\nu}\delta(\xi - \xi_0)$$

the heat conduction equation for $\psi^{(0)}$ and for χ, corresponding to the initial conditions, will be (the common dimensional factor $u_0/(2\nu)$ is henceforth omitted for simplicity)

$$\psi^{(0)}\Big|_{\tau=0} = \partial_\xi\chi\Big|_{\tau=0}, \quad \chi|_{\tau=0} = \frac{(\operatorname{sh}\xi - \operatorname{sh}\xi_0)^\sigma}{\sigma!}H(\xi - \xi_0)$$

They are proportional to the Heaviside function. The solution of these initial problems for the heat-conduction equation enables us to describe the evolution of the concentration field by the following integral representation

$$\begin{aligned} c(\xi,\tau) &= e^{-\sigma^2\tau}\operatorname{ch}\xi\left(\frac{\partial}{\partial\operatorname{sh}\xi}\right)^{\sigma+1}\chi = \\ &= \frac{\exp(-\sigma^2\tau)}{2\sqrt{\pi\tau}}\operatorname{ch}\xi\left(\frac{\partial}{\partial\operatorname{sh}\xi}\right)^{\sigma+1}\int_{\xi_0}^{\infty}\frac{(\operatorname{sh}\varsigma - \operatorname{sh}\xi_0)^\sigma}{\sigma!}\exp\left(-\frac{(\varsigma-\xi)^2}{4\tau}\right)d\varsigma \end{aligned}$$

where the integrals are expressed in terms of the error function, since they can easily be reduced to integrals of the quadratic expressions under the exponential sign.

In particular when $\sigma = 1$ we have

$$c(\xi,\tau) = \frac{\operatorname{sech}^2\xi}{4}[\operatorname{erf}(\lambda+\sqrt{\tau}) - \operatorname{erf}(\lambda-\sqrt{\tau})] + \frac{\exp(-\lambda^2-\tau)\operatorname{ch}\xi_0}{2\sqrt{\pi\tau}\operatorname{ch}\xi}, \quad \lambda \equiv \frac{\xi_0-\xi}{2\sqrt{\tau}}$$

In another special case when $\sigma = 2$ we obtain

$$c(\xi,\tau) = \frac{3}{8}\operatorname{sech}^4\xi\left[\sum_{\pm} \operatorname{erf}(2\sqrt{\tau}\pm\lambda) + 2\exp(-3\tau)\operatorname{sh}\xi\operatorname{sh}\xi_0\sum_{\pm}\operatorname{erf}(\sqrt{\tau}\pm\lambda)\right] +$$
$$+\frac{\exp(-\lambda^2-3\tau)\operatorname{ch}^2\xi_0}{2\sqrt{\pi\tau}\operatorname{ch}^2\xi}$$

The impurity distribution, asymptotically for long times $\tau\to\infty$ and fixed coordinates ξ, ξ_0, here takes the form of a localized soliton-like wave, reflecting the steady solution of the convection equation $c(\xi, \tau) \sim \operatorname{sech}^{2\sigma}\xi$.

Despite the generality (the correspondence when $\sigma = 1$) of the form with solitons of the Korteweg–de Vries equation, the similar impurity solitons differ sharply from the classical ones in the nature of their interactions. When collisions occur there is an absolutely inelastic merging of these asymptotically simplified impurity solitons, in accordance with the inelastic interaction of the shock waves transporting them. As the number of encounters increases we would expect an accumulation of impurity in the solitons. Its concentration will be more pronounced the higher the Prandtl numbers. This may be one of the mechanisms by which coherent structures are formed from shock wave–impurity soliton pairs in turbulent flows of a compressible fluid.

The formation of web-like impurity structures in the field of random two-dimensional compressible flows [17] apparently corresponds to the one-dimensional version considered. The stochastic increase in the mean width of an impurity cluster on the surface of a turbulent flow of fluid was analysed in [18] in the one-dimensional approximation.

3. CONVECTION OF AN ACTIVE IMPURITY

We will now consider the example of an active impurity, which has an inverse effect on the fluid flow, for which it is also possible to reduce the solution of the non-linear equations of motion and convection to the solution of linear heat-conduction equations.

If, due to the effect of the impurity transferred by the flow, the pressure in the liquid varies quadratically with the change in impurity concentration

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = -\partial_x p, \quad p = p_0 + \frac{\gamma}{2}c^2$$

we obtain the following system of linking equations for the velocity and concentration

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = -\gamma c\partial_x c, \quad \partial_t c + \partial_x(uc) - \chi\partial_x^2 c = 0 \tag{3.1}$$

By a simple replacement of the unknowns, consisting of the use of the sum and difference variables $s \equiv u + c\sqrt{\gamma}$, $r \equiv u - c\sqrt{\gamma}$, in the case of equal kinetic coefficients ($\nu = \chi$), we can transfer to a pair of Burgers equations

$$\partial_t s + s\partial_x s - \nu\partial_x^2 s = 0, \quad \partial_t r + r\partial_x r - \nu\partial_x^2 r = 0$$

each of which in turn, by a Cole–Hopf replacement, can be reduced to a linear heat-conduction equation.

We will discuss how this reducibility of the solution of system of equations (3.1) in the case of equal kinetic coefficients to solutions of linear heat-conduction equations is revealed by the Kovalevskaya–Painlevé test. When searching for polar expansions of the solutions [5]

$$u(x,t)=\sum_{n=0}^{\infty} u_n(x,t)\varepsilon^{n-\alpha},\quad c(x,t)=\sum_{n=0}^{\infty} c_n(x,t)\varepsilon^{n-\beta},\quad \varepsilon=\varepsilon(x,t)$$

for the balance of the most singular terms of the equations it is required that $\alpha = \beta = 1$, and Eqs. (3.1) when $\nu = \chi$ are reduced to recurrence relations between the functions $u_n(x, t)$ and $c_n(x, t)$

$$(\partial_t-\nu\partial_x^2)u_{n-2}+(n-2)\Big\{u_{n-1}\varepsilon_t+\frac{\varepsilon_x}{2}\sum_m(u_m u_{n-m}+\gamma c_m c_{n-m})-(n-1)\nu u_n\varepsilon_x^2-$$

$$-\nu\varepsilon_x\partial_x u_{n-1}-\nu\partial_x(\varepsilon_x u_{n-1})\Big\}+\partial_x\frac{1}{2}\sum_m(u_m u_{n-m-1}+\gamma c_m c_{n-m-1})=0$$

$$(\partial_t-\nu\partial_x^2)c_{n-2}+(n-2)\Big\{c_{n-1}\varepsilon_t+\varepsilon_x\sum_m u_m c_{n-m}-$$

$$-(n-1)\nu c_n\varepsilon_x^2-\nu\varepsilon_x\partial_x c_{n-1}-\nu\partial_x(\varepsilon_x c_{n-1})\Big\}+\partial_x\sum_m c_m c_{n-m-1}=0$$

Assuming $n = 0$ we obtain

$$u_0=-\nu\varepsilon_x,\quad c_0=q\frac{\nu}{\sqrt{\gamma}}\varepsilon_x,\quad q=\pm 1 \tag{3.2}$$

The general solution of the system of partial differential equations requires a sufficient number of arbitrary functions (four here), which can be found when the determinant of the coefficients of the leading terms in the recurrent relations vanishes. Using the expressions for u_0 and c_0 this determinant can be converted to the form

$$\Delta=\varepsilon_x^2(n-2)^2\begin{vmatrix} u_0-(n-1)\nu\varepsilon_x & \gamma c_0 \\ c_0 & u_0-(n-1)\nu\varepsilon_x \end{vmatrix}=\nu^2\varepsilon_x^2(n+1)(n-1)(n-2)^2$$

When this vanishes we obtain "resonance" – indices of coefficient functions, which may turn out to be arbitrary.

Using (3.2) the recurrent relations give, when $n = 1$

$$\varepsilon_t-\nu\varepsilon_{xx}+(u_1-q\sqrt{\gamma}c_1)\varepsilon_x=0 \tag{3.3}$$

when $n = 2$

$$\partial_t u_0=\partial_x(\nu\partial_x u_0-u_0u_1-\gamma c_0c_1),\quad \partial_t c_0=\partial_x(\nu\partial_x c_0-u_0c_1-c_0u_1) \tag{3.4}$$

and when $n = 3$

$$\begin{aligned}&(\partial_t+u_1\partial_x-\nu\partial_x^2)u_1+\gamma c_2\partial_x c_1=-\partial_x(u_0u_2+\gamma c_0c_2-\nu\varepsilon_x u_2)-\varepsilon_t u_2-\\&-\varepsilon_x(u_0u_3+u_1u_2+\gamma c_0c_3+\gamma c_1c_2-\nu\partial_x u_2-2\nu\varepsilon_x u_3)\\&(\partial_t-\nu\partial_x^2)c_1+\partial_x(u_1c_1)=-\partial_x(u_0c_2+c_0u_2-\nu\varepsilon_x c_2)-\varepsilon_t c_2-\\&-\varepsilon_x(u_0c_3+u_1c_2+c_0u_3+c_1u_2-\nu\partial_x c_2-2\nu\varepsilon_x c_3)\end{aligned} \tag{3.5}$$

In relation (3.3) one of the functions u_1, c_1 remains undetermined. When $n = 2$, the two relations (3.4), by virtue of (3.2), are a consequence of one another, and moreover are obtained by differentiation of (3.3), so that at this step the functions u_2 and c_2 are not determined. Finally, the required number of arbitrary functions arise necessary to satisfy the Kovalevskaya–Painlevé test: one for any $n = 1$, two for $n = 2$ and one more for $n = -1$ (the latter is $\varepsilon(x, t)$).

It can be seen from the recurrent relations that the series for the solutions can be terminated assuming u_n, $c_n = 0$ when $n > 1$. The functions u_1 and c_1 will then, according to (3.5), satisfy the initial system of equations and formally (3.2)–(3.5) give a Bäcklund transformation, which enables some solutions to be used to construct the others

$$\begin{aligned}&u=\frac{u_0}{\varepsilon}+u_1=-\nu\partial_x\ln\varepsilon+u_1,\quad c=\frac{c_0}{\varepsilon}+c_1=q\frac{\nu}{\sqrt{\gamma}}\partial_x\ln\varepsilon+c_1\\&\varepsilon_t-\nu\varepsilon_{xx}+(u_1-q\sqrt{\gamma}c_1)\varepsilon_x=0\end{aligned} \tag{3.6}$$

If we begin with $u_1 = c_1 = 0$, the solution for u and c turn out to be expressible in terms of the solutions of the heat-conduction equation. The form of the Bäcklund transformation suggests the replacement $v = u - q\sqrt{\gamma c}$. For this variable it takes the form of the Bäcklund transformation for the Burgers equation.

The method employed above of converting to sum and difference variables remains useful when the dispersion of the medium is taken into account, identically for the velocity and impurity fields. For example, in the case of the system of equations

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u + \beta\partial_x^3 u = -\gamma c\partial_x c, \quad \partial_t c + \partial_x(uc) - \nu\partial_x^2 c + \beta\partial_x^3 c = 0$$

this method enables one to change to a pair of Korteweg–de Vries–Burgers equations

$$\partial_t s + s\partial_x s - \nu\partial_x^2 s + \beta\partial_x^3 s = 0, \quad \partial_t r + r\partial_x r - \nu\partial_x^2 r + \beta\partial_x^3 r = 0$$

which reduce to Korteweg–de Vries equations for zero viscosity.

A similar decoupling of the equations is obtained for an arbitrary form of the linear dispersion, which is identical for the velocity and impurity concentration fields.

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